# Exchange economy with quadratic disutility functions

In an economy there are two consumers, $A$ and $B$, who are endowed with two commodities, $x$ and $y$. The preferences of these consumers are given by the following utility functions:

$$u_A = -x_A^2 - y_A^2$$

$$u_B = -x_B^2 - y_B^2$$

Consumer $A$ has an initial endowment $\omega_A = (1, 0)$, while consumer $B$ has an initial endowment $\omega_B = (3, 4)$. Answer the following:

What relative price equilibrates the market? What are the consumption bundles of both consumers after trade?

Even though the statement refers to $x$ and $y$ as goods, the utility functions

$$u_A = -x_A^2 - y_A^2 \qquad u_B = -x_B^2 - y_B^2$$

show that both consumers dislike larger quantities of both commodities. Thus, this is an exchange economy with *bads*

If prices were positive, each consumer would choose

$$(x_i, y_i) = (0, 0)$$

because consuming less of both bads increases utility. Since this cannot clear markets, there is no Walrasian equilibrium with positive prices. Therefore, equilibrium must be supported by *negative prices*

Let

$$p_x = -q_x \qquad p_y = -q_y \qquad q_x > 0,\ q_y > 0$$

Then the relative price is

$$\frac{p_x}{p_y} = \frac{q_x}{q_y}$$

Consumer $A$ solves

$$\max_{x_A, y_A} \ -x_A^2 - y_A^2$$

subject to

$$p_x x_A + p_y y_A \leq p_x \omega_A^x + p_y \omega_A^y$$

Since $\omega_A = (1, 0)$, this becomes

$$-q_x x_A - q_y y_A \leq -q_x$$

or equivalently

$$q_x x_A + q_y y_A \geq q_x$$

So consumer $A$ solves the equivalent minimization problem

$$\min_{x_A, y_A} \ x_A^2 + y_A^2$$

subject to

$$q_x x_A + q_y y_A \geq q_x$$

The Lagrangian is

$$\mathcal{L}_A = x_A^2 + y_A^2 + \lambda_A \left(q_x - q_x x_A - q_y y_A\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_A}{\partial x_A} = 2x_A - \lambda_A q_x = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial y_A} = 2y_A - \lambda_A q_y = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial \lambda_A} = q_x - q_x x_A - q_y y_A = 0$$

From the first two conditions,

$$x_A = \frac{\lambda_A q_x}{2} \qquad y_A = \frac{\lambda_A q_y}{2}$$

Substituting into the constraint,

$$q_x\left(\frac{\lambda_A q_x}{2}\right) + q_y\left(\frac{\lambda_A q_y}{2}\right) = q_x$$

$$\frac{\lambda_A}{2}\left(q_x^2 + q_y^2\right) = q_x$$

$$\lambda_A = \frac{2q_x}{q_x^2 + q_y^2}$$

Hence,

$$x_A = \frac{q_x^2}{q_x^2 + q_y^2} \qquad y_A = \frac{q_x q_y}{q_x^2 + q_y^2}$$

Now consumer $B$ solves

$$\max_{x_B, y_B} \; -x_B^2 - y_B^2$$

subject to

$$p_x x_B + p_y y_B \leq p_x \omega_B^x + p_y \omega_B^y$$

Since $\omega_B = (3, 4)$, this becomes

$$-q_x x_B - q_y y_B \leq -(3q_x + 4q_y)$$

or equivalently

$$q_x x_B + q_y y_B \geq 3q_x + 4q_y$$

Thus, consumer $B$ solves

$$\min_{x_B, y_B} \; x_B^2 + y_B^2$$

subject to

$$q_x x_B + q_y y_B \geq 3q_x + 4q_y$$

The Lagrangian is

$$\mathcal{L}_B = x_B^2 + y_B^2 + \lambda_B\left(3q_x + 4q_y - q_x x_B - q_y y_B\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_B}{\partial x_B} = 2x_B - \lambda_B q_x = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial y_B} = 2y_B - \lambda_B q_y = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial \lambda_B} = 3q_x + 4q_y - q_x x_B - q_y y_B = 0$$

Thus,

$$x_B = \frac{\lambda_B q_x}{2} \qquad y_B = \frac{\lambda_B q_y}{2}$$

Substituting into the constraint,

$$\frac{\lambda_B}{2}\left(q_x^2 + q_y^2\right) = 3q_x + 4q_y$$

$$\lambda_B = \frac{2(3q_x + 4q_y)}{q_x^2 + q_y^2}$$

Hence,

$$x_B = \frac{q_x(3q_x + 4q_y)}{q_x^2 + q_y^2} \qquad y_B = \frac{q_y(3q_x + 4q_y)}{q_x^2 + q_y^2}$$

Now impose market clearing. Total endowments are

$$\bar{x} = 1 + 3 = 4 \qquad \bar{y} = 0 + 4 = 4$$

Thus,

$$x_A + x_B = 4 \qquad y_A + y_B = 4$$

Using the demands above,

$$\frac{q_x^2}{q_x^2 + q_y^2} + \frac{q_x(3q_x + 4q_y)}{q_x^2 + q_y^2} = 4$$

$$\frac{4q_x^2 + 4q_xq_y}{q_x^2 + q_y^2} = 4$$

$$q_x^2 + q_xq_y = q_x^2 + q_y^2$$

$$q_xq_y = q_y^2$$

$$q_x = q_y$$

Therefore,

$$\frac{p_x}{p_y} = \frac{q_x}{q_y} = 1$$

A convenient normalization is

$$p_x = p_y = -1$$

With $q_x = q_y$, the equilibrium bundles are

$$x_A = \frac{1}{2} \qquad y_A = \frac{1}{2}$$

$$x_B = \frac{7}{2} \qquad y_B = \frac{7}{2}$$

So the equilibrium allocation is

$$(x_A, y_A) = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad (x_B, y_B) = \left(\frac{7}{2}, \frac{7}{2}\right)$$

Hence, the market-clearing relative price is $\frac{p_x}{p_y} = 1$, supported by negative prices such as $p_x = p_y = -1$, and the equilibrium allocation is $\left(\frac{1}{2}, \frac{1}{2}\right)$ for $A$ and $\left(\frac{7}{2}, \frac{7}{2}\right)$ for $B$